Marginal cost derivations for a Cobb-Douglas production function

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Imagine a producer wants to minimise costs from labour L_t and capital K_t, which come at factor prices w_t and r_t, respectively, subject to a production technology of the Cobb-Douglas function $Y_t = A_t L_t^{1-\alpha} K_t^{\alpha}$, where A_t is total factor productivity and α the capital share of production. The minimal cost is given as

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t, K_t} w_t L_t + r_t K_t \text{ s.t. } Y_t = A_t L_t^{1-\alpha} K_t^{\alpha}.$$

Solving the constraint for capital, we obtain

$$K_t = \left(\frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}},$$

so that

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t} w_t L_t + r_{k,t} \left(\frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}}.$$

The first-order condition of that problem is

$$\begin{aligned} w_t &= \frac{1-\alpha}{\alpha} r_{k,t} \left(\frac{Y_t}{A_t L_t} \right)^{\frac{1}{\alpha}} \\ &= \frac{1-\alpha}{\alpha} r_{k,t} \left(\frac{K_t}{L_t} \right)^{\frac{1}{\alpha}}, \end{aligned} \tag{1}$$

so the optimal use of labour in production, L_t^*, is given by

$$L_t^*(w_t, r_t, Y_t, A_t) = \left(\frac{1-\alpha}{\alpha} \frac{r_t}{w_t} \right)^{\alpha} \frac{Y_t}{A_t}.$$

Putting this back into the constraint, we obtain the optimal use of capital in production, K_t^*, as

$$K_t^*(w_t, r_t, Y_t, A_t) = \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{1-\alpha} \frac{Y_t}{A_t}.$$

Now plugging $L_t^*$and K_t^* into the initial minimisation problem, we obtain

$$\begin{aligned} C(w_t, r_t, Y_t, A_t) &= \left[\left(\frac{1-\alpha}{\alpha} \frac{r_t}{w_t} \right)^{\alpha} w_t + \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \right)^{1-\alpha} r_t \right] \frac{Y_t}{A_t} \\ &= \left[\left(\frac{1-\alpha}{\alpha} \right)^{\alpha} + \left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} \right] w_t^{1-\alpha} r_t^{\alpha} \frac{Y_t}{A_t} \\ &= \left[\frac{1-\alpha+\alpha}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right] w_t^{1-\alpha} r_t^{\alpha} \frac{Y_t}{A_t} \\ &= \left(\frac{r_t}{\alpha} \right)^{\alpha} \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha} \frac{Y_t}{A_t} \end{aligned}$$

The marginal cost is then just the first derivative of w.r.t. output Y_t:

$$MC_t = \left(\frac{r_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \frac{1}{A_t}. \tag{2}$$

Note that an equivalent way of expressing (1) and (2) is to use the first-order conditions of a competitive wholesale producer charging the price $P_{w,t} = MC_t$, i.e. who solves the problem

$$\max_{L_t, K_t} P_{w,t} A_t L_t^{1-\alpha} K_t^{\alpha} - w_t L_t - r_t K_t,$$

The problem yields the conditions

$$r_t = P_{w,t} \alpha A_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{3}$$

$$w_t = P_{w,t} (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{1-\alpha}. \tag{4}$$

To see that they are equivalent to the above derivations, divide (3) by (4) to obtain (1) and rearrange

$$\left(\frac{r_t}{\alpha} \frac{1}{P_{w,t} A_t}\right)^{\frac{1}{1-\alpha}} \overset{(3)}{=} \frac{L_t}{K_t} \overset{(4)}{=} \left(\frac{1-\alpha}{w_t} P_{w,t} A_t\right)^{\frac{1}{\alpha}},$$

which together with $P_{w,t} = MC_t$ gives (2). It is only important to only use only two out of the equations, because otherwise equations become collinear. Examples for the use of (1) and (2) is Fernández-Villaverde and Rubio-Ramírez (2006), while two for the use of (3) and (4), with no derivation of $P_{w,t}$, are Gertler and Karadi (2011) and Abbritti and Fahr (2013).

References

Abbritti, M. and S. Fahr (2013). Downward wage rigidity and business cycle asymmetries. *Journal of Monetary Economics 60*(7), 871–886.

Fernández-Villaverde, J. and J. F. Rubio-Ramírez (2006). A baseline dsge model. Technical report, Duke University.

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