Marginal cost derivations for a Cobb-Douglas production function

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Imagine a producer wants to minimise costs from labour $L_t$ and capital $K_t$, which come at factor prices $w_t$ and $r_t$, respectively, subject to a production technology of the Cobb-Douglas function $Y_t = A_t L_t^{1-\alpha} K_t^\alpha$, where $A_t$ is total factor productivity and $\alpha$ the capital share of production. The minimal cost is given as

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t, K_t} w_t L_t + r_t K_t \text{ s.t. } Y_t = A_t L_t^{1-\alpha} K_t^\alpha.$$  

Solving the constraint for capital, we obtain

$$K_t = \left( \frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}},$$

so that

$$C(w_t, r_t, Y_t, A_t) = \min_{L_t} w_t L_t + r_t \left( \frac{Y_t}{A_t L_t^{1-\alpha}} \right)^{\frac{1}{\alpha}}.$$  

The first-order condition of that problem is

$$w_t = \frac{1 - \alpha}{\alpha} r_{k,t} \left( \frac{Y_t}{A_t L_t} \right)^{\frac{1}{\alpha}}$$

$$= \frac{1 - \alpha}{\alpha} r_{k,t} \left( \frac{K_t}{L_t} \right)^{\frac{1}{\alpha}},$$  

so the optimal use of labour in production, $L_t^*$, is given by

$$L_t^*(w_t, r_t, Y_t, A_t) = \left( \frac{1 - \alpha}{\alpha} \frac{r_t}{w_t} \right)^{\frac{1}{\alpha}} \frac{Y_t}{A_t}.$$  

Putting this back into the constraint, we obtain the optimal use of capital in production, $K_t^*$, as

$$K_t^*(w_t, r_t, Y_t, A_t) = \left( \frac{\alpha}{1 - \alpha} \frac{w_t}{r_t} \right)^{1-\frac{1}{\alpha}} \frac{Y_t}{A_t}.$$  

Now plugging $L_t^*$ and $K_t^*$ into the initial minimisation problem, we obtain

$$C(w_t, r_t, Y_t, A_t) = \left[ \left( \frac{1 - \alpha}{\alpha} \frac{r_t}{w_t} \right)^{\alpha} \frac{w_t}{A_t} + \left( \frac{\alpha}{1 - \alpha} \frac{w_t}{r_t} \right)^{1-\alpha} r_t \right] \frac{Y_t}{A_t}$$

$$= \left[ \frac{1 - \alpha}{\alpha} \right] + \left( \frac{\alpha}{1 - \alpha} \right)^{1-\alpha} \frac{w_t^{1-\alpha} r_t^\alpha Y_t}{A_t}$$

$$= \frac{r_t}{\alpha} \left( \frac{w_t}{1 - \alpha} \right)^{1-\alpha} \frac{Y_t}{A_t}.$$  

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The marginal cost is then just the first derivative of w.r.t. output $Y_t$:

$$MC_t = \left(\frac{r_t}{\alpha}\right)^\alpha \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \frac{1}{A_t}. \quad (2)$$

Note that an equivalent way of expressing (1) and (2) is to use the first-order conditions of a competitive wholesale producer charging the price $P_{w,t} = MC_t$, i.e. who solves the problem

$$\max_{L_t,K_t} P_{w,t} A_t L_t^{1-\alpha} K_t^{\alpha} - w_t L_t - r_t K_t,$$

The problem yields the conditions

$$r_t = P_{w,t} \alpha A_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} \quad (3)$$

$$w_t = P_{w,t} (1 - \alpha) A_t \left(\frac{K_t}{L_t}\right)^{1-\alpha}. \quad (4)$$

To see that they are equivalent to the above derivations, divide (3) by (4) to obtain (1) and rearrange

$$\left(\frac{r_t}{\alpha P_{w,t} A_t}\right)^{\frac{1}{1-\alpha}} \equiv L_t^{\frac{\alpha}{1-\alpha}} \left(\frac{1-\alpha}{w_t} P_{w,t} A_t\right)^{\frac{1}{1-\alpha}},$$

which together with $P_{w,t} = MC_t$ gives (2). It is only important to only use only two out of the equations, because otherwise equations become collinear. Examples for the use of (1) and (2) is Fernández-Villaverde and Rubio-Ramírez (2006), while two for the use of (3) and (4), with no derivation of $P_{w,t}$, are Gertler and Karadi (2011) and Abbritti and Fahr (2013).

**References**

